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On the Signatures of the Canonical and the Metrical Energy Densities of Vector Fields

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Abstract

The energy-density components $\Theta^0{}_0$ and $T^0{}_0$ of the canonical and of the metrical energy-momentum tensors $\Theta^i{}_k$ and $T^i{}_k$ for a statical field of vector mesons have opposite signatures: $\Theta^0{}_0 = H = -T^0{}_0 = -L$. From this property some relativistic and field-theoretical theorems can be deduced in an elementary way.

The Lagrangian of a vector meson field is (cf: Wentzel, 1949; Hund, 1954)

$$L = \tfrac{1}{4}F_{ik}F^{ik} + \tfrac{1}{2}k^2 A_i A^i = \tfrac{1}{2}(F_{0\nu}F^{0\nu} + \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + k^2 A^i A_i) \tag{1}$$

with $i = 0, 1, 2, 3$; $\nu = 1, 2, 3$ and with the field tensor

$$F_{ik} = A_{k,i} - A_{i,k}, \qquad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \qquad F_{0\nu} = A_{\nu,0} - A_{0,\nu} \tag{1a}$$

For a statical vector field we have

$$A^\nu = A_\nu = 0 \qquad (\nu = 1, 2, 3), \qquad A^0 = A_0 = \varphi(X^\lambda), \qquad \varphi_{,0} = 0 \tag{2}$$

With (2) the Lagrangian (1) becomes the negative definite expression

$$L = -\tfrac{1}{2}(\varphi_{,\nu}\varphi_{,\nu} + k^2\varphi^2) \qquad (\varphi = A_0) \tag{3}$$

The canonical energy tensor to the Lagrangian (1) is

$$\Theta^i{}_k = (\partial L/\partial A^l{}_{,i})A^l{}_{,k} - \delta^i{}_k L = F^i{}_l F_k{}^l + F^{il}A_{k,l} - \delta^i{}_k L$$

$$\Theta_{ik} = F_{il}A^l{}_{,k} - \eta_{ik}L \tag{4}$$

and the metrical energy tensor defined by Hilbert (1924) is

$$T_{ik} = (2/\sqrt{-g})[\delta(\sqrt{-g}L)/\delta g^{ik}] = F_{il}F_k{}^l + k^2 A_i A_k - \eta_{ik}L \qquad (5)$$

In (4) and (5)

$$\eta_{ik} = \text{diagonal}(-1, +1, +1, +1)$$

is the Minkowski tensor.

For the statical field (2) the spacelike components of Θ_{ik} and T_{ik} become identical:

$$\Theta_{\mu\nu} = T_{\mu\nu} = -\varphi_{,\mu}\varphi_{,\nu} + \tfrac{1}{2}\delta_{\mu\nu}(\varphi_{,\lambda}\varphi_{,\lambda} + k^2\varphi^2) \qquad (\mu, \nu = 1, 2, 3) \qquad (6a)$$

and the space-time components of both tensors vanish:

$$T_{\mu 0} = \Theta_{\mu 0} = \Theta_{0\mu} = 0 \qquad (6b)$$

But the timelike energy component of Θ_{ik} is given by the negative definite expression[1]

$$\Theta_{00} = -\tfrac{1}{2}(\varphi_{,\nu}\varphi_{,\nu} + k^2\varphi^2) = L \leqslant 0, \qquad \Theta^0{}_0 = H = -L \qquad (7)$$

and the energy component of T_{ik} is given by the positive definite expression

$$T_{00} = \varphi_{,\nu}\varphi_{,\nu} + k^2\varphi^2 + L = \tfrac{1}{2}(\varphi_{,\nu}\varphi_{,\nu} + k^2\varphi^2) = -L \geqslant 0 \qquad (8)$$

Therefore, for a statical vector field the energy densities defined by the canonical and by the metrical tensor have opposite signatures:[2]

$$T_{00} = -\Theta_{00} = -L = \tfrac{1}{2}(\varphi_{,\nu}\varphi_{,\nu} + k^2\varphi^2) \qquad (9)$$

This point makes clear an old problem discussed by Laue (1953): In a bimetrical theory of gravitation the energy density $\mathfrak{t}^{\,i}_{00}$ of a statical field becomes positive

[1] The Hamiltonian to the Lagrangian (1) (cf. Wentzel, 1949) follows from

$$\begin{aligned}\Theta_0{}^0 = -\Theta_{00} = H &= (\partial L/\partial A^l{}_{,0})A^l{}_{,0} - L\\ &= F^{0\nu}F_{0\nu} + F^{0\nu}A_{0,\nu} - L\\ &= \tfrac{1}{2}(F^{0\nu}F_{0\nu} - \tfrac{1}{2}F^{\mu\nu}F_{\mu\nu} - k^2A_iA^i) + F^{0\nu}A_{0,\nu}\\ &= -\tfrac{1}{2}\pi_\nu\pi_\nu + \pi_\nu A_{0,\nu} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}k^2A_iA^i\end{aligned}$$

with the canonical momentum

$$\pi_i = \partial L/\partial A^i{}_{,0} = F^0{}_i = -F_{0i} = F_{i0}, \qquad \pi_0 = 0$$

[2] With (2) the equations (6)–(9) are valid for a field A_i with the Lagrangian of Wentzel (1949):

$$L^* = \tfrac{1}{2}(A^l{}_{,i}A_{l,k}\eta^{ik} + k^2A_iA^i), \qquad \pi^*{}_i = -A_{i,0}$$

and for Fermi's version of electrodynamics (cf. Wentzel, (1949), too.

definite in spite of the fact, that this gravitation field describes an attractive interaction. Laue has defined the energy tensor density $\mathfrak{t}^i{}_k$ of the bimetrical gravitation field in analogy to Einstein's general relativistic affine tensor

$$\mathfrak{t}^i{}_k = \tfrac{1}{2}[(\partial\mathscr{L}/\partial g_{mn,i})g_{mn,k} - \delta^i{}_k\mathscr{L}] \tag{10}$$

with the Lagrangian $\mathscr{L}$ for the gravitation field g_{ik}. From (10) as the energy density of a statical field the positive definite expression (cf. Laue, 1953)

$$\mathfrak{t}_{00} = -\tfrac{1}{2}g_{00}\mathscr{L} \geqslant 0 \tag{11}$$

results, because for statical g_{ik} the Lagrange density $\mathscr{L}$ is positive definite, and it is $g_{00} < 0$. $\mathscr{L} \geqslant 0$ means that the statical gravitation interaction gives attractive forces.

However, Laue's tensor $\mathfrak{t}_{ik}$ is a canonical energy tensor, and the canonical energy density Θ_{00} of a statical vector field (2) is negative definite

$$\Theta_{00} = L \leqslant 0$$

because this L is a Lagrangian of a repulsive interaction.

For source-free vector fields, that is, for a closed field theory, with the field equations

$$\delta L/\delta A^i = F_i{}^l{}_{,l} + k^2 A_i = 0 \qquad (\rightarrow A^i{}_{,i} = 0 \quad \text{for } k \neq 0) \tag{12}$$

the connection between the metrical and the canonical tensor is given by the divergence of a superpotential (cf. Hund 1954). According to (12) we have

$$T_{ik} = \Theta_{ik} + H_{ik}{}^l{}_{,l} \tag{13}$$

with the superpotential

$$H_{ikl} = -H_{lki} = A_k F_{li} \tag{13a}$$

and with the conservation laws

$$T^i{}_{k,i} = \Theta^i{}_{k,i} = H^i{}_k{}^l{}_{,li} = 0 \tag{13b}$$

For a time-independent source-free field A_i we have with $A_{i,0} = 0$

$$T_{ik} = \Theta_{ik} + H_{ik\nu,\nu} \tag{14}$$

and the space integrals over these T_{ik} can be written

$$\int_{V_3} T_{ik}\, d^3x = \int_{V_3} \Theta_{ik}\, d^3x + \int_{\delta V_3} H_{ik\nu}\, dS^\nu \qquad (\nu = 1, 2, 3) \tag{15}$$

according to the Gaussian theorem. Therefore, for time-independent fields A_i without sources and without singularities the volume integrals over the metrical and the canonical tensor are equivalent, if V_3 is the space x^0 = const; it is according to (12):

$$\int_{x^0=\text{const}} T_{ik}\, d^3x = \int_{x^0=\text{const}} \Theta_{ik}\, d^3x \tag{16}$$

From (16) and (9) it follows, that for a source-free statical vector field (2) the energy integral must vanish:

$$\int_{x^0=\text{const}} T_{00}\, d^3x = \int_{x^0=\text{const}} \Theta_{00}\, d^3x = -\int_{x^0=\text{const}} \Theta_{00}\, d^3x$$

$$= -\int_{x^0=\text{const}} L\, d^3x = 0 \tag{17}$$

and because of the definity of the statical L this function must vanish itself:

$$-2L = \varphi_{,\nu}\varphi_{,\nu} + k^2\varphi^2 = 0 \tag{18}$$

Equation (18) implies well-known potential-theoretical theorems for statical fields without sources and singularities. Equation (18) gives in the case of vector mesons (with $k \neq 0$)

$$\varphi_{,\nu} = \varphi_{,i} = 0 \quad \text{and} \quad \varphi = 0 \rightarrow T_{ik} = \Theta_{ik} = 0 \tag{19a}$$

and for the electrostatical field (with $k = 0$)

$$\varphi_{,\nu} = \varphi_{,i} = 0 \rightarrow T_{ik} = \Theta_{ik} = 0 \tag{19b}$$

That means, a statical source-free field of vector mesons vanishes, $A_i = 0$, and a source-free electrostatical field has a vanishing field strength, $F_{ik} = 0$.[3]

Generally, in a nonclosed theory of vector fields (cf. Hund, 1954) a source-density current s_i defined by

$$\delta L/\delta A^i = F_i^{\ l}{}_{,l} + k^2 A_i = s_i \tag{20}$$

exists with the charge density $-s^0 = s_0 = \rho$. Then the connection between the canonical energy tensor (4) and the metrical energy tensor (5) is given by

$$\begin{aligned} T_{ik} &= \Theta_{ik} + H_{ik}{}^{l}{}_{,l} - A_k F^l{}_{i,l} \\ &= \Theta_{ik} + (A_k F^l{}_i)_{,l} + A_k F_i{}^l{}_{,l} \\ &= \Theta_{ik} + H_{ik}{}^{l}{}_{,l} + A_k s_i \end{aligned} \tag{21}$$

[3] That the source-free stationary field A^i vanishes, too, follows from this theorem in a simple way: The time-independent field equations (12) are

$$-A^i{}_{,\mu\mu} + k^2 A^i = -(\Delta - k^2)A^i = 0$$

But, the same equation $(\Delta - k^2)\varphi = 0$ defines the source-free potential $A^0 = -\varphi$ in the statical case for which we have proved that $\varphi = 0$ is the only solution without singularities.

For a time-independent vector field with sources the integral relations

$$\int_{x^0=\text{const}} T_{ik}\, d^3x = \int_{x^0=\text{const}} \Theta_{ik}\, d^3x + \int_{x^0=\text{const}} A_k s_i\, d^3x$$

follow. For a statical vector field (2) we have $s_\nu = 0$, and the equations (6a) and (6b) are valid. But, for the energy-density components (9) the integral relation

$$\int_{x^0=\text{const}} T_{00} = - \int_{x^0=\text{const}} \Theta_{00}\, d^3x = \int_{x^0=\text{const}} \Theta_{00}\, d^3x + \int_{x^0=\text{const}} \varphi\rho\, d^3x \tag{23}$$

results from (22). Equation (23) determines the interaction energy of a statical vector field to

$$\int T_{00}\, d^3x = \frac{1}{2}\int (\varphi_{,\nu}\varphi_{,\nu} + k^2\varphi^2)\, d^3x = \int \Theta^0{}_0\, d^3x = \frac{1}{2}\int \rho\varphi\, d^3x \tag{24}$$

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